Trigonometric Functions & Proofs Review (Part 1)

Trigonometric Functions & Proofs Review — Student Solution Sheet

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1. Coterminal Angles

a)

$$\frac{23\pi}{4} - \frac{7\pi}{4} = \frac{16\pi}{4} = 4\pi = 2\pi(2)$$

The angles are coterminal.
General formula:

$$\theta = \frac{7\pi}{4} + 2\pi n$$

---

b)

$$\frac{13\pi}{6} - \left(-\frac{11\pi}{6} \right) = \frac{24\pi}{6} = 4\pi = 2\pi(2)$$

The angles are coterminal.
General formula:

$$\theta = -\frac{11\pi}{6} + 2\pi n$$

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2. Coterminal Angle Properties

a)

$$\sin A = \sin B$$

Always True
Coterminal angles terminate at the same point on the unit circle.

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b)

$$\tan A = \tan B$$

Always True
Coterminal angles have identical sine and cosine values.

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c)

$$A - B = 2\pi n$$

Always True
This is the definition of coterminal angles.

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3. Exact Values

a)

$$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

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b)

$$\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

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c)

$$\tan\left(\frac{11\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

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4. Given sin $\boldsymbol{\theta = -\frac{5}{13}}$

Since

$$x^2 + y^2 = 1$$

and

$$y = -\frac{5}{13}$$

then

$$x^2 + \left(\frac{5}{13}\right)^2 = 1$$
$$x^2 + \frac{25}{169} = 1$$
$$x^2 = \frac{144}{169}$$
$$x = \pm\frac{12}{13}$$

a) Coordinates

$$\left(\frac{12}{13}, -\frac{5}{13}\right)$$
$$\left(-\frac{12}{13}, -\frac{5}{13}\right)$$

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b)

$$\cos\ \theta = \pm\frac{12}{13}$$

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c)

$$\tan\ \theta = \frac{y}{x}$$
$$\tan\ \theta = -\frac{5}{12}$$

or

$$\tan\ \theta = \frac{5}{12}$$

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d)

Possible quadrants:

$$\boxed{\text{III and IV}}$$

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5. Reciprocal Functions

a)

$$\csc\left(\frac{\pi}{6}\right) = \frac{1}{\sin(\pi/6)} = \frac{1}{1/2} = 2$$

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b)

$$\sec\left(\frac{2\pi}{3}\right) = \frac{1}{-1/2} = -2$$

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c)

$$\cot\left(\frac{5\pi}{4}\right) = \frac{1}{\tan\,(5\pi/4)} = \frac{1}{1} = 1$$

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6. Is it Possible?

a)

$$\csc\,\theta = -3$$

Possible

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b)

$$\sin\,\theta = 1.2$$

Impossible

$$-1 \leq \sin\,\theta \leq 1$$

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c)

$$\cot\,\theta = 0$$

Possible

when $\cos\,\theta = 0$

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d)

$$\sec\,\theta = 0.5$$

Impossible

$$\cos\,\theta = 2$$

which cannot occur.

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7. Solve on $\mathbf{0 \leq \theta < 2\pi}$

a)

$$\sin\,\theta = \frac{1}{2}$$

Reference angle:

$$\frac{\pi}{6}$$

Solutions:

$$\boxed{\frac{\pi}{6}, \frac{5\pi}{6}}$$

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b)

$$\cos\,\theta = -\frac{\sqrt{2}}{2}$$

Reference angle:

$$\frac{\pi}{4}$$

Solutions:

$$\boxed{\frac{3\pi}{4}, \frac{5\pi}{4}}$$

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8. Solve

$$3\sin^2\theta + \sin\theta - 2 = 0$$

Factor:

$$(3\sin\theta - 2)(\sin\theta + 1) = 0$$

Case 1

$$\sin\theta = \frac{2}{3}$$
$$\theta \approx 0.730$$
$$\theta \approx 2.412$$

Case 2

$$\sin\theta = -1$$
$$\theta = \frac{3\pi}{2}$$

Solutions

$$\boxed{\theta \approx 0.730,\ 2.412,\ \frac{3\pi}{2}}$$

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9. Solve

$$3\sin\theta = 2$$
$$\sin\theta = \frac{2}{3}$$
$$\theta \approx 41.8^\circ$$
$$180^\circ - 41.8^\circ = 138.2^\circ$$
$$\boxed{41.8^\circ,\ 138.2^\circ}$$

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10. Does a Solution Exist?

$$\sec\theta = 0.8$$
$$\cos\theta = \frac{1}{0.8} = 1.25$$

Since cosine cannot exceed 1:

$$\boxed{\text{No solution}}$$

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11. $y = -\cos(2x)$

Amplitude:

$$\boxed{1}$$

Period:

$$\boxed{\pi}$$

Quarter-period:

$$\boxed{\frac{\pi}{4}}$$

Five key points:

$$(0,-1)$$
$$\left(\frac{\pi}{4},0\right)$$
$$\left(\frac{\pi}{2},1\right)$$
$$\left(\frac{3\pi}{4},0\right)$$
$$(\pi,-1)$$

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12. Comparing $y = \sin(2x)$ and $y = -\cos(2x)$

Amplitude:

$$1 \text{ for both}$$

Period:

$$\pi \text{ for both}$$

Cycles on $0 \le x \le 2\pi$:

$$2 \text{ for both}$$

Starting behaviour:

$$\sin(2x) \to (0,0)$$
$$-\cos(2x) \to (0,-1)$$

Transformations:

- horizontal compression by $\frac{1}{2}$
- cosine graph also reflected across the x-axis

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13. Solve

$$\sin x = \cos x$$
$$\tan x = 1$$
$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Coordinates:

$$\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right)$$
$$\left(\frac{5\pi}{4},-\frac{\sqrt{2}}{2}\right)$$

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14. Exactly One Intersection

$$y = \sin x$$

intersects a horizontal line exactly once only at its maximum or minimum.

$$\boxed{k = 1 \text{ or } k = -1}$$

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15. $y = -4\sin\left(x - \frac{\pi}{3}\right) + 2$

Amplitude:

$$4$$

Midline:

$$y = 2$$

Maximum:

$$6$$

Minimum:

$$-2$$

Range:

$$\boxed{-2 \le y \le 6}$$

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16. Why No x-Intercepts?

$$y = 2\sin\, x + 5$$

Amplitude:

$$2$$

Midline:

$$5$$

Range:

$$3 \le y \le 7$$

The graph never reaches $y = 0$.
Therefore:

$$\boxed{\text{No x-intercepts}}$$

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17. $y = 3\cos\left(x - \frac{\pi}{2}\right) + 1$

Period:

$$2\pi$$

Quarter-period:

$$\frac{\pi}{2}$$

First key x-value:

$$\frac{\pi}{2}$$

Last key x-value:

$$\frac{5\pi}{2}$$

Five key points:

$$\left(\frac{\pi}{2}, 4\right)$$
$$(\pi, 1)$$
$$\left(\frac{3\pi}{2}, -2\right)$$
$$(2\pi, 1)$$
$$\left(\frac{5\pi}{2}, 4\right)$$

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18. Amplitude 5, Midline −2, Shift Right $\frac{\pi}{4}$

Possible equation:

$$y = 5\sin\left(x - \frac{\pi}{4}\right) - 2$$

Five key points:

$$\left(\frac{\pi}{4}, -2\right)$$
$$\left(\frac{3\pi}{4}, 3\right)$$
$$\left(\frac{5\pi}{4}, -2\right)$$
$$\left(\frac{7\pi}{4}, -7\right)$$
$$\left(\frac{9\pi}{4}, -2\right)$$

19. Period and Number of Cycles

a)

$$y = \sin(3\theta)$$

Period:

$$\frac{2\pi}{3}$$

Number of cycles on $0 \le \theta \le 2\pi$:

$$\frac{2\pi}{2\pi/3} = 3$$

$$\boxed{\text{Period } = \frac{2\pi}{3}, 3 \text{ cycles}}$$

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b)

$$y = \cos\left(\frac{\theta}{2}\right)$$

Period:

$$\frac{2\pi}{1/2} = 4\pi$$

Number of cycles on $0 \le \theta \le 2\pi$:

$$\frac{2\pi}{4\pi} = \frac{1}{2}$$

$$\boxed{\text{Period } = 4\pi, \frac{1}{2} \text{ cycle}}$$

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20. Number of x-Intercepts

$$y = \sin(5\theta)$$

Period:

$$\frac{2\pi}{5}$$

Number of cycles:

$$\frac{2\pi}{2\pi/5} = 5$$

A sine graph has zeros at the start, middle, and end of each cycle, but neighbouring cycles share endpoints.

Solve directly:

$$\sin(5\theta) = 0$$
$$5\theta = n\pi$$
$$\theta = \frac{n\pi}{5}$$

On $0 \leq \theta \leq 2\pi$:

$$n = 0,1,2,\ldots,10$$

So there are:

$$\boxed{\text{11 x-intercepts}}$$

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21. Does the Function Have x-Intercepts?

$$y = 3\sin(2\theta) + 5$$

Amplitude:

$$3$$

Midline:

$$y = 5$$

Maximum:

$$5 + 3 = 8$$

Minimum:

$$5 - 3 = 2$$

Range:

$$2 \leq y \leq 8$$

Since $y = 0$is not in the range:

$$\boxed{\text{No x-intercepts}}$$

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22. Full Graph Analysis

$$y = -4\sin\left(3\theta - \frac{\pi}{2}\right) + 2$$

Rewrite the inside:

$$3\theta - \frac{\pi}{2} = 3\left(\theta - \frac{\pi}{6}\right)$$

So:

$$y = -4\sin\left(3\left(\theta - \frac{\pi}{6}\right)\right) + 2$$

Amplitude:

$$\boxed{4}$$

Period:

$$\frac{2\pi}{3}$$

Number of cycles:

$$\frac{2\pi}{2\pi/3} = 3$$

Maximum:

$$2 + 4 = 6$$

Minimum:

$$2 - 4 = -2$$

Range:

$$\boxed{-2 \leq y \leq 6}$$

Now find the x-intercepts.

$$0 = -4\sin\left(3\theta - \frac{\pi}{2}\right) + 2$$
$$-2 = -4\sin\left(3\theta - \frac{\pi}{2}\right)$$
$$\sin\left(3\theta - \frac{\pi}{2}\right) = \frac{1}{2}$$

Let:

$$u = 3\theta - \frac{\pi}{2}$$

Then:

$$\sin u = \frac{1}{2}$$
$$u = \frac{\pi}{6} + 2\pi n$$

or

$$u = \frac{5\pi}{6} + 2\pi n$$

First family

$$3\theta - \frac{\pi}{2} = \frac{\pi}{6} + 2\pi n$$
$$3\theta = \frac{\pi}{6} + \frac{\pi}{2} + 2\pi n$$
$$3\theta = \frac{4\pi}{6} + 2\pi n$$
$$3\theta = \frac{2\pi}{3} + 2\pi n$$
$$\theta = \frac{2\pi}{9} + \frac{2\pi n}{3}$$

Second family

$$3\theta - \frac{\pi}{2} = \frac{5\pi}{6} + 2\pi n$$
$$3\theta = \frac{5\pi}{6} + \frac{\pi}{2} + 2\pi n$$
$$3\theta = \frac{8\pi}{6} + 2\pi n$$
$$3\theta = \frac{4\pi}{3} + 2\pi n$$
$$\theta = \frac{4\pi}{9} + \frac{2\pi n}{3}$$

On $0 \leq \theta \leq 2\pi$, the x-intercepts are:

$$\boxed{\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}}$$

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23. Write a Cosine Function

Maximum:

$$8$$

Minimum:

$$2$$

Amplitude:

$$\frac{8-2}{2}=3$$

Midline:

$$\frac{8+2}{2}=5$$

Phase shift:

$$\frac{\pi}{3}\text{ right}$$

A possible function is:

$$\boxed{y=3\cos\left(x-\frac{\pi}{3}\right)+5}$$

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24. Two Equivalent Trigonometric Equations

One possible pair is:

$$y=\sin x$$

and

$$y=\cos\left(x-\frac{\pi}{2}\right)$$

Cosine shifted right by $\frac{\pi}{2}$produces the sine graph.

Therefore:

$$\boxed{\sin x=\cos\left(x-\frac{\pi}{2}\right)}$$

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25. Daylight Model Information

Maximum daylight:

$$15$$

Minimum daylight:

$$9$$

a) Amplitude

$$\frac{15-9}{2}=3$$

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b) Midline

$$\frac{15+9}{2}=12$$

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26. Ferris Wheel Model

Maximum height:

$$54$$

Minimum height:

$$6$$

Period:

$$20$$

The amplitude is:

$$\frac{54-6}{2}=24$$

The midline is:

$$\frac{54+6}{2}=30$$

Since the period is 20:

$$B=\frac{2\pi}{20}$$
$$B=\frac{\pi}{10}$$

The rider starts at the lowest point, so use negative cosine.

$$\boxed{h(t)=-24\cos\left(\frac{\pi}{10}t\right)+30}$$

Now determine when the rider is above 40 m.

$$-24\cos\left(\frac{\pi}{10}t\right)+30>40$$
$$-24\cos\left(\frac{\pi}{10}t\right)>10$$

Divide by -24, reversing the inequality:

$$\cos\left(\frac{\pi}{10}t\right)<-\frac{5}{12}$$

Solve the boundary points:

$$\cos\left(\frac{\pi}{10}t\right)=-\frac{5}{12}$$
$$\frac{\pi}{10}t=\cos^{-1}\left(-\frac{5}{12}\right)$$
$$\frac{\pi}{10}t\approx 1.999$$
$$t\approx 6.4$$

The second time in the cycle is:

$$2\pi-1.999\approx 4.284$$
$$\frac{\pi}{10}t\approx 4.284$$
$$t\approx 13.6$$

Therefore, the rider is above 40 mfor:

$$\boxed{6.4<t<13.6}$$

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27. Identify the Identity Family

a)

$$\sec^2\theta-1$$

Use the Pythagorean identity:

$$\sec^2\theta-1=\tan^2\theta$$
$$\boxed{\text{Pythagorean identity}}$$

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b)

$$1 - \sin^2 \theta$$

Use the Pythagorean identity:

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$\boxed{\text{Pythagorean identity}}$$

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c)

$$\tan \theta \cdot \cot \theta$$

Use reciprocal identities:

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\tan \theta \cdot \cot \theta = 1$$

$$\boxed{\text{Reciprocal identity}}$$

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28. Simplify Completely

$$\frac{\tan \theta + \cot \theta}{\sec \theta \cdot \csc \theta}$$

Rewrite each function using sine and cosine:

$$\frac{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}}{\frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta}}$$

Simplify the numerator:

$$\frac{\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}}{\frac{1}{\sin \theta \cos \theta}}$$

Use the Pythagorean identity:

$$\sin^2 \theta + \cos^2 \theta = 1$$

So:

$$\frac{\frac{1}{\sin \theta \cos \theta}}{\frac{1}{\sin \theta \cos \theta}} = 1$$

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29. Prove

$$(\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = 1$$

Start with the left side:

$$(\sec \theta - \tan \theta)(\sec \theta + \tan \theta)$$

Use difference of squares:

$$\sec^2 \theta - \tan^2 \theta$$

Use the identity:

$$\sec^2 \theta = 1 + \tan^2 \theta$$

Substitute:

$$1 + \tan^2 \theta - \tan^2 \theta = 1$$

Therefore:

$$\boxed{(\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = 1}$$

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30. Prove

$$\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$$

Start with the left side:

$$\sec^2 x + \csc^2 x$$

Rewrite using sine and cosine:

$$\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}$$

Use a common denominator:

$$\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}$$
$$\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}$$

Use the Pythagorean identity:

$$\sin^2 x + \cos^2 x = 1$$
$$\frac{1}{\sin^2 x \cos^2 x}$$

Rewrite:

$$\frac{1}{\sin^2 x} \cdot \frac{1}{\cos^2 x}$$
$$\csc^2 x \sec^2 x$$
$$\boxed{\sec^2 x \csc^2 x}$$

Therefore:

$$\boxed{\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x}$$

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31. Identify the Single Function

$$\cos 70^\circ \cos 20^\circ + \sin 70^\circ \sin 20^\circ$$

Use the identity:

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

So:

$$\cos 70^\circ \cos 20^\circ + \sin 70^\circ \sin 20^\circ$$
$$= \cos(70^\circ - 20^\circ)$$
$$= \cos 50^\circ$$

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32. Exact Value

$$\sin\left(\frac{5\pi}{12}\right)$$

Rewrite the angle:

$$\frac{5\pi}{12} = \frac{3\pi}{12} + \frac{2\pi}{12}$$
$$\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$$

So:

$$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

Use:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

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33. Double-Angle Values

Given:

$$\sin\theta = \frac{3}{5}$$

and $\theta$is in Quadrant I.

Use a 3-4-5 triangle:

$$\cos\theta = \frac{4}{5}$$

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a)

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$$

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b)

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

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34. Double-Angle Identities

a)

$$2\sin(3x)\cos(3x)$$

Use:

$$2\sin A\cos A = \sin(2A)$$

Here:

$$A = 3x$$

So:

$$2\sin(3x)\cos(3x) = \sin(6x)$$

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b)

$$\cos^2(2x) - \sin^2(2x)$$

Use:

$$\cos^2 A - \sin^2 A = \cos(2A)$$

Here:

$$A = 2x$$

So:

$$\cos^2(2x) - \sin^2(2x) = \cos(4x)$$

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c)

Both answers use **double-angle identities**.

In part a:

$$2\sin\ A\cos\ A = \sin\ (2A)$$

changes an expression with angle Ainto a single function with angle $2A$.

In part b:

$$\cos^2 A - \sin^2 A = \cos\ (2A)$$

also changes an expression with angle Ainto a single function with angle $2A$.

So both expressions are connected because they simplify using double-angle identities.